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Mathematical analysis of a thermo-visco-plastic model with Bodner–Partom constitutive equations

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ABSTRACT

We study a thermo-mechanical model, where the mechanical model of inelastic deformation due to S.R. Bodner and Y. Partom is coupled with a heat equation. The main result is the existence and uniqueness of the solution to the thermo-visco-plastic model.

1. Introduction

We consider a thermo-visco-plastic model of deformation. This model describes phenomena in a deformed metal additionally exposed to a heat modification. The considered model complements the mechanical problem by temperature. On the one hand the temperature "controls" the domain of an elastic behaviour of the body (by the function β and a thermal part of the stress), on the other hand the strain and the stress appearing in the body influence the heat conduction (when the body is deformed its temperature can change). The mechanical part (without the temperature function) of the considered model was proposed in the seventies, in a less general version, by S.R. Bodner and Y. Partom in [1]. About ten years ago K. Chełmiński and P. Gwiazda [2,3,5] proposed a class of functions $\mathcal{G}$, γ and δ (including the functions from the original constitutive Bodner–Partom equations), for which they have shown existence and uniqueness of solutions. In this paper we couple the equations describing the mechanical model with a heat conduction equation. We assume that the considered body initially occupies a bounded domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary $\partial\Omega$. Further, $x \in \Omega$ is a material point, while $t \in (0, +\infty)$ represents time. The coupled system reads

$$\begin{aligned}
&-\operatorname{div}_x \sigma(x,t) = f(x,t),\\
&\sigma(x,t) = \mathcal{D}\big(\varepsilon\big(\nabla_x u(x,t)\big) - \varepsilon^p(x,t)\big) - c\mathbb{I}\big(\theta(x,t) - \theta_0(x)\big),\\
&\partial_t \varepsilon^p(x,t) = \mathcal{G}\left(\frac{\{|\sigma^D(x,t)| + \beta(\theta(x,t))\}_+}{y(x,t)}\right)\frac{\sigma^D(x,t)}{|\sigma^D(x,t)|},\\
&\partial_t y(x,t) = \gamma\big(y(x,t)\big)\mathcal{G}\left(\frac{|\sigma^D(x,t)|}{y(x,t)}\right)\big|\sigma^D(x,t)\big| - A\delta\big(y(x,t)\big),\\
&\partial_t\theta(x,t) = \kappa\Delta\theta(x,t) - \alpha \operatorname{div}_x \partial_t u(x,t) + \partial_t\varepsilon^p(x,t)\cdot\sigma(x,t).
\end{aligned} \tag{TBP}$$

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The first equation is the balance of momentum. The function $\sigma : \Omega \times \mathbb{R}_+ \to \mathcal{S}^3$ is the stress tensor. We denote by $\mathcal{S}^3$ the set of symmetric 3×3-matrices with real entries. The given vector function $f : \Omega \times \mathbb{R}_+ \to \mathbb{R}^3$ is the density of external forces. In the second equation a relation between the stress σ and a displacement function $u : \Omega \times \mathbb{R}_+ \to \mathbb{R}^3$ is given. The symmetric part of the displacement $\varepsilon(\nabla_x u) = \frac{1}{2}(\nabla_x u + \nabla_x^T u)$ is the Cauchy strain tensor in the case of a small deformation. It follows from the definition that it is also an element of $\mathcal{S}^3$. The second equation is a generalisation of Hooke's law connecting the stress, elastic part of the Cauchy strain tensor, and the temperature function $\theta : \Omega \times \mathbb{R}_+ \to \mathbb{R}$. The operator $\mathcal{D} : \mathcal{S}^3 \to \mathcal{S}^3$ is linear, symmetric and positive definite. The third equation describes the evolution of the plastic part of the strain tensor in time. We denote by σ^D the stress deviator, that is $\sigma^D = \sigma - \frac{1}{3}\operatorname{tr}\sigma \cdot \mathbb{I}$, where $\mathbb{I}$ denotes a second order unit tensor. The given function $\mathcal{G} : \mathbb{R}_+ \to \mathbb{R}_+$ is a generalisation of the function of the plastic constitutive equation proposed by Bodner and Partom, and we denote $x_+ = \max\{0, x\}$. The given function $\beta : \mathbb{R} \to \mathbb{R}$ describes the influence of temperature on the plastic behaviour of the material. An internal variable $y : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ describes the isotropic hardening of the metal. The evolution of this variable is modelled by the fourth formula. In that relation $\gamma : \mathbb{R}_+ \supset D(\gamma) \to \mathbb{R}_+$ and $\delta : \mathbb{R}_+ \supset D(\delta) \to \mathbb{R}_+$ are given functions, while $A \geqslant 0$ is constant. The system is completed by the heat conduction equation. Moreover c, κ and α are positive constants that depend on the material. We consider our problem with Dirichlet boundary condition

$$\begin{aligned} u(x,t) &= g_{\mathcal{D}}(x,t) \quad \text{for } (x,t) \in \partial\Omega \times \mathbb{R}_+, \\ \theta(x,t) &= g_\theta(x,t) \quad \text{for } (x,t) \in \partial\Omega \times \mathbb{R}_+, \end{aligned} \tag{D}$$

and set the initial condition a

$$\begin{aligned} \varepsilon^p(x,0) &= \varepsilon_0^p(x), \\ y(x,0) &= y_0(x), \\ \theta(x,0) &= \theta_0(x). \end{aligned} \tag{I}$$

Moreover we assume that ε_0^p is a symmetric and traceless matrix for all $x \in \Omega$. Our aim is to find a solution $(u, \varepsilon^p, y, \theta)$ to the system above and prove its uniqueness. In the first part of this paper we will prove the existence of the solution using a Galerkin approximation. First we will cancel the boundary data by solving a linear problem. Then we will construct a Galerkin approximative sequence for the homogeneous problem and use energy estimates to show that it is bounded in $H^1((0,T) \times \Omega)$. Further, using a compactness argument, we will prove convergence of the Galerkin approximation. In the second part of this paper we will show the uniqueness of the solution to the considered problem. The uniqueness will be proved by analysing the difference of two other solutions, with the need of stronger assumptions.

Assumptions

We formulate the first group of assumptions for a nonlinear constitutive function $\mathcal{G}$:

A1 $\mathcal{G}(p)$ and $\frac{\mathcal{G}(p)}{p}$ are $\mathbb{C}^\infty(\mathbb{R}_+, \mathbb{R}_+)$ functions,
A2 There exists $g \in \mathbb{R}$ such that $\mathcal{G}(p) < g$ for all $p \in \mathbb{R}_+$ ($\mathcal{G}$ is bounded),
A3 $\mathcal{G}'(p) \geqslant 0$ for all $p \in \mathbb{R}_+$ ($\mathcal{G}$ is nondecreasing),
A4 There exists g' such that $\mathcal{G}'(p)p < g'$ for all $p \in \mathbb{R}_+$.

The second group of assumptions concerns the functions γ and δ:

A5 $D(\gamma) = D(\delta) = [y_2, y_1]$, where $0 < y_2 < y_1$ and γ, δ are $\mathbb{C}^\infty(D(\gamma), \mathbb{R}_+)$ functions,
A6 $\gamma'(p) \leqslant 0$ for all $p \in [y_2, y_1]$ (γ is nonincreasing),
A7 $\gamma(y_2) > \gamma(y_1) = 0, \delta(y_2) = 0$ moreover $\delta(p) > 0$ for all $p \in (y_2, y_1]$.

The last group of assumptions is related to the function β describing the coupling of the heat equation and the inelastic constitutive equation:

A8 β is a $\mathbb{C}^\infty(\mathbb{R}, \mathbb{R})$ function,
A9 There exists $k > 0$ such that $\beta(t) \in (-k, 0)$ for all $t \in \mathbb{R}$,
A10 $\lim_{t\to\infty} \beta(t) = 0$,
A11 There exists B such that $B \geqslant \beta'(t) \geqslant 0$ for all $t \in \mathbb{R}$.

Assumptions **A1**–**A7** have been taken from article [5]. Additionally we have formulated assumptions **A8**–**A11** which are connected with the extension of the system by the heat equation.

Remark 1. S.R. Bodner and Y. Partom have proposed (see [1]) functions $\mathcal{G}$, γ and δ in the form

$$\mathcal{G}(p) = d\exp\left(\frac{-\alpha}{p^n}\right),$$
$$\gamma(y) = m(y_1 - y),$$
$$\delta(y) = y_1\left(\frac{y - y_2}{y_1}\right)^r,$$

where $n, r > 1$, $d, m > 0$, $y_1 > y_2 > 0$ are constants depending on the considered material and $\alpha = 1/2 + 1/n$. It is easy to see that the function $\mathcal{G}$ from above satisfies assumptions **A1**–**A4** and functions γ and δ satisfy assumptions **A5**–**A7**.

Remark 2. We will denote $T(x,t) = \mathcal{D}(\varepsilon(\nabla_x u(x,t)) - \varepsilon^p(x,t))$. We can easily see that

$$\varepsilon^p \cdot \sigma = \varepsilon^p \cdot \sigma^D = \varepsilon^p \cdot T = \varepsilon^p \cdot T^D,$$

where $T^D = T - \frac{1}{3}\operatorname{tr} T \cdot \mathbb{I}$, because σ as well as T are symmetric matrices and ε^p is traceless almost everywhere provided $\operatorname{tr}\varepsilon_0^p(x) = 0$ for almost all $x \in \Omega$. In fact $\sigma(x,t) - T(x,t)$ is a diagonal matrix for almost all $(x,t) \in \Omega \times \mathbb{R}_+$. Moreover from (TBP3) we immediately see that $\operatorname{tr}\partial_t\varepsilon^p = 0$ so $\operatorname{tr}\varepsilon^p = 0$. Additionally we denote $F(x,t) := f(x,t) + c\nabla_x\theta_0(x)$.

Remark 3. We denote by $\|\cdot\|$ the standard norm in the space $L^2(\Omega)$ (with $(\cdot,\cdot)$ being the underlying inner product) and by $\|\cdot\|_\eta$ the norm in the Sobolev space $H^\eta(\Omega)$.

Remark 4. The initial value $u(x,0)$ can immediately be calculated from Eq. (TBP1) with given data ε_0^p, θ_0 and $g_{\mathcal{D}}(x,0)$ (see for example [4]).

We are going to use the energy method to prove the existence of a solution to (TBP). So we define the free energy function (the potential energy) for the triple (u, ε^p, y):

$$\mathcal{E}\big(u,\varepsilon^p,y\big)(t) = \frac{1}{2}\big(T(t), \varepsilon\big(\nabla_x u(t)\big) - \varepsilon^p(t)\big) = \frac{1}{2}\big(\mathcal{D}\big(\varepsilon\big(\nabla_x u(t)\big) - \varepsilon^p(t)\big), \varepsilon\big(\nabla_x u(t)\big) - \varepsilon^p(t)\big).$$

One can easily observe, using properties of the operator $\mathcal{D}$, that we can find a constant $\omega > 0$ such that

$$\mathcal{E}\big(u,\varepsilon^p,y\big)(t) \geqslant \omega\big\|\varepsilon\big(\nabla_x u(t)\big) - \varepsilon^p(t)\big\|^2.$$

2. Existence of a solution to (TBP)

2.1. Transformation to a homogeneous boundary-value problem

Similarly as in article [5] we remove the boundary values and the function of the force density by a suitable transformation. To this end we consider the following pair of linear equations. The first one is

$$\partial_t\tilde{\theta}(x,t) = \kappa\Delta\tilde{\theta}(x,t), \tag{LP1}$$

and the second is

$$\begin{aligned} -\operatorname{div}_x \tilde{T}(x,t) &= -\nabla_x\tilde{\theta} + F(x,t),\\ \tilde{T}(x,t) &= \mathcal{D}\big(\varepsilon\big(\nabla_x\tilde{u}(x,t)\big)\big). \end{aligned} \tag{LP2}$$

With the equations above we relate Dirichlet boundary conditions of the form:

$$\tilde{\theta}(x,t)|_{\partial\Omega} = g_\theta(x,t)|_{\partial\Omega},$$
$$\tilde{u}(x,t)|_{\partial\Omega} = g_{\mathcal{D}}(x,t)|_{\partial\Omega}$$

and smooth initial conditions:

$$\tilde{\theta}(x,0) = \tilde{\theta}_0(x)$$

compatible with the boundary data

$$\tilde{\theta}_0(x)|_{\partial\Omega} = g_\theta(x,0)|_{\partial\Omega}.$$

Lemma 1.

a) *Let the function* $g_\theta \in L^2(0,T^*;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))$ *with* $\partial_t g_\theta \in L^2(0,T^*;L^2(\Omega))$. *Moreover let the initial data* $\tilde{\theta}_0$ *have* $H^1(\Omega)$ *regularity and satisfy the compatibility condition above. Then the problem* (LP1) *has a unique solution* $\tilde{\theta}$ *satisfying* $\tilde{\theta} \in L^2(0,T^*;H^2(\Omega)) \cap L^\infty(0,T^*;H^1(\Omega))$ *with* $\partial_t\tilde{\theta} \in L^2(0,T^*;L^2(\Omega))$ *and*

$$\begin{aligned}&\|\tilde{\theta}\|_{L^2(0,T^*;H^2(\Omega))} + \|\tilde{\theta}\|_{L^\infty(0,T^*;H^1(\Omega))} + \|\partial_t\tilde{\theta}\|_{L^2(0,T^*;L^2(\Omega))}\\ &\quad\leqslant C\big(T^*\big)\big(\|\tilde{\theta}_0\|_{H^1(\Omega)} + \|g_\theta\|_{L^2(0,T^*;H^2(\Omega))} + \|g_\theta\|_{L^\infty(0,T^*;H^1(\Omega))} + \|\partial_t g_\theta\|_{L^2(0,T^*;L^2(\Omega))}\big).\end{aligned}$$

b) *Let* $F \in L^\infty(0,T^*;L^2(\Omega)) \cap H^1(0,T^*;L^2(\Omega))$ *and* $g_{\mathcal{D}} \in L^\infty(0,T^*;H^2(\Omega)) \cap W^{1,\infty}(0,T^*;H^1(\Omega))$. *Then the problem* (LP2) *has a unique solution* $\tilde{u}$ *belonging to* $W^{1,\infty}(0,T^*;H^2(\Omega))$ *and we have the estimate*

$$\begin{aligned}&\|\tilde{u}\|_{L^\infty(0,T^*;H^2(\Omega))} \leqslant C\big(\|\tilde{\theta}\|_{L^\infty(0,T^*;H^1(\Omega))} + \|F\|_{L^\infty(0,T^*;L^2(\Omega))} + \|g_{\mathcal{D}}\|_{L^\infty(0,T^*;H^2(\Omega))}\big),\\ &\|\partial_t\tilde{u}\|_{L^\infty(0,T^*;H^1(\Omega))} \leqslant C\big(T^*\big)\big(\|\partial_t\tilde{\theta}\|_{L^2(0,T^*;L^2(\Omega))} + \|\partial_t F\|_{L^2(0,T^*;L^2(\Omega))}\\ &\qquad + \|\partial_t g_{\mathcal{D}}\|_{L^\infty(0,T^*;H^1(\Omega))}\|\tilde{\theta}_0\|_{L^2(\Omega)} + \big\|F(0)\big\|_{L^2(\Omega)}\big).\end{aligned}$$

The first part of the lemma above is the standard result for parabolic equations, the second part is an obvious consequence of the ellipticity of the operator $\tilde{u} \mapsto \operatorname{div}_x \mathcal{D}(\varepsilon(\nabla_x\tilde{u}))$.

Now we can transform system (TBP) into a homogeneous one. Let $\hat{u} := u - \tilde{u}$, $\hat{T} := T - \tilde{T}$ and $\hat{\theta} := \theta - \tilde{\theta}$ where u and θ are solutions for the problem (TBP). We can easily see that "hat" functions satisfy

$$\begin{aligned}&\operatorname{div}_x \hat{T}(x,t) = c\nabla_x\hat{\theta}(x,t),\\ &\hat{T}(x,t) = \mathcal{D}\big(\varepsilon\big(\nabla_x\hat{u}(x,t)\big) - \varepsilon^p(x,t)\big),\\ &\partial_t\varepsilon^p(x,t) = \mathcal{G}\bigg(\frac{\{|T^D(x,t)| + \beta(\theta(x,t))\}_+}{y(x,t)}\bigg)\frac{T^D(x,t)}{|T^D(x,t)|},\\ &\partial_t y(x,t) = \gamma\big(y(x,t)\big)\mathcal{G}\bigg(\frac{|T^D(x,t)|}{y(x,t)}\bigg)\big|T^D(x,t)\big| - A\delta\big(y(x,t)\big),\\ &\partial_t\hat{\theta}(x,t) = \kappa\,\Delta\hat{\theta}(x,t) - \alpha\operatorname{div}_x\partial_t u(x,t) + \partial_t\varepsilon^p(x,t)\cdot T(x,t),\end{aligned} \tag{HP}$$

with homogeneous Dirichlet boundary condition, and with the initial data:

$$\begin{aligned}&\hat{u}(x,0) = u(x,0) - \tilde{u}_0(x) =: \hat{u}_0(x),\\ &\hat{\theta}(x,0) = \theta_0(x) - \tilde{\theta}_0(x) =: \hat{\theta}_0(x),\\ &\varepsilon^p(x,0) = \varepsilon^p_0(x),\\ &y(x,0) = y_0(x),\end{aligned}$$

where we obtain $u(x,0)$ according to Remark 4.

Remark 5. It is easy to see that ε^p and y are the same in (TBP) and (HP), since these functions are not present in (LP1) and (LP2): we can thus rewrite Eqs. (TBP3) and (TBP4) into the system (HP). For a similar reason we have functions u, T and ε^p without hats on the right-hand side of Eq. (HP5). This means that the displacement and the stress tensor are composed of known functions $\tilde{u}$, $\tilde{T}$ and unknown functions $\hat{u}$, $\hat{T}$.

2.2. Galerkin approximation of HP

We are going to approximate a solution of (HP) by the Galerkin method. The procedure is similar to the one used in [2,3,5,6]. We will regularise the singular behaviour of the function $\mathcal{G}(p)/p^2$ at the point $p = 0$. Thus in a manner similar to [5] we construct a sequence of functions $\{\mathcal{G}_k\}_{k=1}^\infty$ acting from $\mathbb{R}_+$ to $\mathbb{R}_+$. First we employ the cut-off function $\chi : \mathbb{R}_+ \to [0,1]$ of class C^∞ such that $\chi(p) = 0$ for $p < \frac{1}{2}$ and $\chi(p) = 1$ for $p > 1$. Moreover let χ be nondecreasing and $p\chi'(p) \leqslant \chi_0$ for all $p \geqslant 0$ where $\chi_0 \in \mathbb{R}_+$ is a constant. Now we define $\mathcal{G}_k$ by $\mathcal{G}_k(p) := \chi(kp)\mathcal{G}(p)$. We can check (see [2,3]) that the sequence of functions $\{\mathcal{G}_k\}_{k=1}^\infty$ satisfies assumptions **A1**–**A4**. In particular

$$0 \leqslant \mathcal{G}_k(p) = \chi(kp)\mathcal{G}(p) \leqslant \mathcal{G}(p) \leqslant g =: g_k.$$

Moreover we can conclude from assumptions on the function χ that

$$\mathcal{G}_k'(p)p = kp\chi'(kp)\mathcal{G}(p) + \chi(kp)\mathcal{G}'(p)p \leqslant \chi_0\mathcal{G}\left(\frac{1}{k}\right) + g' =: g_k'.$$

Next we use assumptions **A1**–**A2** on $\mathcal{G}$ to obtain $g_k' \to g'$ when $k \to \infty$, with the estimate $g_k' \leqslant \chi_0 g + g'$. Furthermore, it follows directly from the definition of $\mathcal{G}_k$ that $\mathcal{G}_k \rightrightarrows \mathcal{G}$ (where $\rightrightarrows$ denotes the uniform convergence of functions) when $k \to \infty$. Indeed

$$\sup_{p\geqslant 0}\left|\mathcal{G}_k(p) - \mathcal{G}(p)\right| = \sup_{\frac{1}{k}\geqslant p\geqslant 0}\left|\chi(kp)\mathcal{G}(p) - \mathcal{G}(p)\right| = \sup_{\frac{1}{k}\geqslant p\geqslant 0}\left|\chi(kp) - 1\right|\left|\mathcal{G}(p)\right| \leqslant 2\mathcal{G}\left(\frac{1}{k}\right).$$

In the k-th step of the Galerkin procedure we replace $\mathcal{G}$ by $\mathcal{G}_k$. To obtain the sequence of smooth functions $\{\hat{u}_k, \varepsilon_k^p, y_k, \hat{\theta}_k\}_{k=1}^{\infty}$ we need also that the function F, the boundary data $g_{\mathcal{D}}$, g_θ and the initial conditions ε_0^p, y_0, θ_0, $\hat{\theta}_0$ are C^∞ functions.

Let $\{v_k\}_{k=1}^{\infty}$ be a sequence of eigenvectors associated with the eigenvalues $0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots$ of the operator $\mathcal{L}: H^2(\Omega,\mathbb{R}^3) \cap H_0^1(\Omega,\mathbb{R}^3) \to L^2(\Omega,\mathbb{R}^3)$ defined by $\mathcal{L}v = -\operatorname{div}_x \mathcal{D}(\varepsilon(\nabla_x v))$. Moreover let $\{w_k\}_{k=1}^{\infty}$ be a sequence of eigenvectors associated with eigenvalues $0 < \eta_1 \leqslant \eta_2 \leqslant \eta_3 \leqslant \cdots$ of the operator $-\Delta: H^2(\Omega,\mathbb{R}) \cap H_0^1(\Omega,\mathbb{R}) \to L^2(\Omega,\mathbb{R})$. It follows from the properties of operators $\mathcal{L}$ and $-\Delta$ that $\{v_k\}_{k=1}^{\infty} \subset C^\infty(\Omega,\mathbb{R}^3)$ forms an orthonormal system in the space $L^2(\Omega,\mathbb{R}^3)$ and an orthogonal basis in the space $H_0^1(\Omega,\mathbb{R}^3)$ while $\{w_k\}_{k=1}^{\infty} \subset C^\infty(\Omega,\mathbb{R})$ is an orthonormal system in the space $L^2(\Omega,\mathbb{R})$ and orthogonal basis in the space $H_0^1(\Omega,\mathbb{R})$. We set $\hat{u}_k$ and $\hat{\theta}_k$ as a finite linear combination

$$\hat{u}_k(x,t) = \sum_{j=1}^{k} \phi_k^j(t) v_j(x),$$

$$\hat{\theta}_k(x,t) = \sum_{j=1}^{k} \psi_k^j(t) w_j(x),$$

where ϕ_j^k, ψ_j^k as well as ε_k^p, y_k are found by solving the following system for each $j = 1, 2, \ldots, k$:

$$\begin{aligned}
&(\hat{T}_k, \nabla_x v_j) = -c(\nabla_x\hat{\theta}_k, v_j),\\
&\hat{T}_k = \mathcal{D}\big(\varepsilon(\nabla_x\hat{u}_k) - \varepsilon_k^p\big),\\
&\partial_t\varepsilon_k^p = \mathcal{G}_k\left(\frac{\{|T_k^D| + \beta(\theta_k)\}_+}{y_k}\right)\frac{T_k^D}{|T_k^D|},\\
&\partial_t y_k = \gamma(y_k)\mathcal{G}_k\left(\frac{|T_k^D|}{y_k}\right)\left|T_k^D\right| - A\delta(y_k),\\
&(\partial_t\hat{\theta}_k, w_j) + \kappa(\nabla_x\hat{\theta}_k, \nabla_x w_j) = \alpha(\partial_t\hat{u}_k, \nabla_x w_j) - \alpha(\operatorname{div}_x \partial_t\tilde{u}, w_j) + \big(\partial_t\varepsilon_k^p T_k, w_j\big),
\end{aligned} \tag{GA}$$

where we denote $T_k := \hat{T}_k - \tilde{T}$. The system (GA) is considered with the initial data

$$\begin{aligned}
&\phi_j^k(0) = (\hat{u}_0, v_j),\\
&\psi_j^k(0) = (\hat{\theta}_0, w_j),\\
&\varepsilon_k^p(x,0) = \varepsilon_0^p(x),\\
&y_k(x,0) = y_0(x).
\end{aligned} \tag{IG}$$

To prove existence and uniqueness of the solution to the problem (GA) we need the following lemma to ensure that we can choose the basis $\{v_k\}_{k=1}^{\infty} \subset H_0^1(\Omega,\mathbb{R}^3)$ such that the matrix $\{(v_i, \nabla w_j)\}_{i,j=1}^{m}$ is invertible.

Lemma 2. *Let $\{v_k\}_{k=1}^{\infty}$ be any basis in $H_0^1(\Omega,\mathbb{R}^3)$ and $\{w_k\}_{k=1}^{\infty}$ be any basis in $H_0^1(\Omega,\mathbb{R})$. Then there exists a permutation $p(i)$ of the basis $\{v_k\}_{k=1}^{\infty}$ such that for all $m \in \mathbb{N}$ the matrix $\{(v_{p(i)}, \nabla_x w_j)\}_{i,j=1}^{m}$ is nonsingular.*

The lemma above was proven in [7].

Proposition 1. *Assume that the solutions $\tilde{\theta}$ to* (LP1) *and $\tilde{u}$ to* (LP2) *are $C^\infty((0,T^*) \times \Omega)$ functions. Moreover let the initial conditions be smooth, i.e. $\varepsilon_0^p \in C^\infty(\Omega)$, $y_0 \in C^\infty(\Omega)$, $\theta_0 \in C^\infty(\Omega)$, and $\hat{\theta}_0 \in C^\infty(\Omega)$. Then for all $k = 1, 2, \ldots$ there exists the unique local-in-time solution $(\hat{u}_k, \varepsilon_k^p, y_k, \hat{\theta}_k)$ to* (GA)*.*

Outline of the proof. Using properties of the chosen basis we can rewrite system (GA) in the following form

$$
\begin{aligned}
&\lambda_j\phi_k^j = \big(\mathcal{D}\varepsilon_k^p, \nabla_x v_j\big) - c\sum_{l=1}^{k}\psi_k^l(\nabla_x w_l, v_j),\\
&\hat{T}_k = \sum_{j=1}^{k}\big(\phi_k^j\mathcal{D}\big(\varepsilon(\nabla_x w_j)\big)\big) - \mathcal{D}\big(\varepsilon_k^p\big),\\
&\partial_t\varepsilon_k^p = \mathcal{G}_k\bigg(\frac{\{|T_k^D| + \beta(\theta_k)\}_+}{y_k}\bigg)\frac{T_k^D}{|T_k^D|},\\
&\partial_t y_k = \gamma(y_k)\mathcal{G}_k\bigg(\frac{|T_k^D|}{y_k}\bigg)\big|T_k^D\big| - A\delta(y_k),\\
&\partial_t\psi_k^j + \kappa\sum_{l=1}^{k}\psi_k^l\eta_l = \alpha\sum_{l=1}\partial_t\phi_k^l(v_l, \nabla_x w_j) - \alpha(\operatorname{div}_x\partial_t\tilde{u}, w_j) + \big(\partial_t\varepsilon_k^p T_k, w_j\big).
\end{aligned}
\tag{GA*}
$$

Using Lemma 2, we can uniquely solve the linear problem (GA*1), (GA*2), and (GA*3) with initial data (IG1) and (IG2) for fixed $\varepsilon_k^p \in C^\infty([0,T^*]\times\Omega)$ and $y_k \in C^\infty([0,T^*]\times\Omega)$ satisfying (IG3) and (IG5) respectively. For the obtained sequences $\{\phi_k^j\}_{j=1}^k \subset C^\infty([0,T_k^*])$ and $\{\psi_k^j\}_{j=1}^k \subset C^\infty([0,T_k^*])$ we set

$$
\begin{aligned}
&\bar{\varepsilon}_k^p(t) = \int_0^t \mathcal{G}_k\bigg(\frac{\{|T_k^D| + \beta(\theta_k)\}_+}{y_k}\bigg)\frac{T_k^D}{|T_k^D|}\,ds + \varepsilon_0^p,\\
&\bar{y}_k(t) = \int_0^t \gamma(y_k)\mathcal{G}_k\bigg(\frac{|T_k^D|}{y_k}\bigg)\big|T_k^D\big| - A\delta(y_k)\,ds + y_0.
\end{aligned}
$$

It is easy to see that $\bar{\varepsilon}_k^p \in C^\infty([0,T_k^*]\times\Omega)$ and $\bar{y}_k \in C^\infty([0,T_k^*]\times\Omega)$. Now we can choose such a short time interval $[0,T_k^*]$ such that the operator $\mathcal{P} : (\varepsilon_k^p, y_k) \mapsto (\bar{\varepsilon}_k^p, \bar{y}_k)$ is a contraction. Applying the Banach fixed point theorem completes the proof. $\square$

Remark 6. Slight modification of Lemma 1 leads us to the observation that the solutions $\tilde{\theta}$ to (LP1) and $\tilde{u}$ to (LP2) are $C^\infty((0,T^*)\times\Omega)$ functions provided that F, $g_{\mathcal{D}}$, and g_θ are smooth functions.

Remark 7. We show only the local-in-time solution of the k-th step of the Galerkin approximation. From energy estimates proven in the subsequent sections it follows that the solution is global in time, i.e. $T_k^* = \infty$.

2.3. Energy estimate

First we recall Lemma 2.1 from [5]. This result is important for energy estimates and gives us a uniform bound for the isotropic hardening y.

Lemma 3. *If we assume that the function σ^D is continuous then for all C^1 solutions $y(x,t)$ of Eq.* (TBP4) *with the initial data satisfying $y_2 \leqslant y_0(x) \leqslant y_1$ we have the estimate $y_2 \leqslant y(x,t) \leqslant y_1$ for all $(x,t) \in \Omega\times\mathbb{R}_+$.*

Lemma 4. *The energy function $\mathcal{E}(t)$ defined for the sequence $(\hat{u}_k, \varepsilon_k^p, y_k)$ is uniformly bounded in time, moreover for all $T^* \geqslant 0$ the sequence $\hat{\theta}_k$ is bounded in $L^\infty(0,T^*;L^2(\Omega))$ and the sequence $\nabla_x\hat{\theta}_k$ is bounded in $L^2(0,T^*;L^2(\Omega))$. Additionally for all $t \geqslant 0$ the following inequality holds*:

$$
\begin{aligned}
&\mathcal{E}\big(\hat{u}_k, \varepsilon_k^p, y_k\big)(t) + \big\|\hat{\theta}_k(t)\big\|^2 + \int_0^t \big\|\nabla_x\hat{\theta}_k(\tau)\big\|^2\,d\tau + \int_0^t\int_\Omega \frac{\mathcal{G}_k(\frac{\{|T_k^D|+\beta(\theta_k)\}_+}{y_k})}{|T_k^D|}\big|\hat{T}_k^D\big|^2\,dx\,d\tau\\
&\quad\leqslant C(t)\bigg(\mathcal{E}\big(\hat{u}_k, \varepsilon_k^p, y_k\big)(0) + \big\|\hat{\theta}_k(0)\big\|^2 + \int_0^t \big\|\tilde{T}^D\big\|^2\,d\tau + \int_0^t \|\operatorname{div}_x\partial_t\tilde{u}\|^2\,d\tau\bigg).
\end{aligned}
$$

Proof. In the k-th step of the Galerkin approximation we multiply Eq. (GA1) by $\partial_t\phi_k^j$ and sum over $j=1,\dots,k$ to obtain

$$\big(\hat{T}_k, \varepsilon(\partial_t\nabla_x\hat{u}_k)\big) - c(\hat{\theta}_k, \operatorname{div}_x\partial_t\hat{u}_k) = 0,$$

then we subtract the term $(\hat{T}_k, \partial_t\varepsilon_k^p)$

$$\big(\hat{T}_k, \varepsilon(\partial_t\nabla_x\hat{u}_k) - \partial_t\varepsilon_k^p\big) - \frac{c}{\alpha}(\hat{\theta}_k, \alpha\operatorname{div}_x\partial_t\hat{u}_k) = -\big(\hat{T}_k, \partial_t\varepsilon_k^p\big).$$

Analogously we use the approximated heat equation (GA5) to obtain

$$(\partial_t\hat{\theta}_k, \hat{\theta}_k) + \kappa(\nabla_x\hat{\theta}_k, \nabla_x\hat{\theta}_k) = \alpha(\partial_t\hat{u}_k, \nabla_x\hat{\theta}_k) - \alpha(\operatorname{div}_x\partial_t\tilde{u}, \hat{\theta}_k) + \big(\partial_t\varepsilon_k^p T_k, \hat{\theta}_k\big).$$

Considering the relations $(T_k, \varepsilon_k^p) = (T_k^D, \varepsilon_k^p)$ and the definition of the energy function we get

$$\partial_t\mathcal{E}\big(\hat{u}_k, \varepsilon_k^p, y_k\big)(t) + \frac{c\kappa}{\alpha}(\nabla_x\hat{\theta}_k, \nabla_x\hat{\theta}_k) + \frac{c}{\alpha}(\hat{\theta}_k, \partial_t\hat{\theta}_k) = \frac{c}{\alpha}\big(\hat{\theta}_k, \partial_t\varepsilon_k^p T_k^D\big) - \big(\hat{T}_k, \partial_t\varepsilon_k^p\big) + c(\hat{\theta}_k, \operatorname{div}_x\partial_t\tilde{u}).$$

We can simplify the second and the third terms on the left-hand side

$$\partial_t\mathcal{E}\big(\hat{u}_k, \varepsilon_k^p, y_k\big)(t) + \frac{c\kappa}{\alpha}\|\nabla_x\hat{\theta}_k\|^2 + \frac{c}{2\alpha}\partial_t\|\hat{\theta}_k\|^2 = \frac{c}{\alpha}\big(\hat{\theta}_k, \partial_t\varepsilon_k^p T_k^D\big) - \big(\hat{T}_k, \partial_t\varepsilon_k^p\big) + c(\hat{\theta}_k, \operatorname{div}_x\partial_t\tilde{u}).$$

Let us now consider the term $-(\hat{T}_k, \partial_t\varepsilon_k^p)$ and use Eq. (GA3):

$$\begin{aligned}
-\big(\hat{T}_k, \partial_t\varepsilon_k^p\big) &= -\int_\Omega \mathcal{G}_k\bigg(\frac{\{|T_k^D| + \beta(\theta_k)\}_+}{y_k}\bigg)\frac{T_k^D}{|T_k^D|}\hat{T}_k^D\,dx \\
&= -\int_\Omega \frac{\mathcal{G}_k(\frac{\{|T_k^D|+\beta(\theta_k)\}_+}{y_k})}{|T_k^D|}\big|\hat{T}_k^D\big|^2\,dx - \int_\Omega \frac{\mathcal{G}_k(\frac{\{|T_k^D|+\beta(\theta_k)\}_+}{y_k})}{|T_k^D|}\tilde{T}^D\hat{T}_k^D\,dx \\
&\leqslant -\frac{1}{2}\int_\Omega \frac{\mathcal{G}_k(\frac{\{|T_k^D|+\beta(\theta_k)\}_+}{y_k})}{|T_k^D|}\big|\hat{T}_k^D\big|^2\,dx + \frac{1}{2y_2}\sup_{p\geqslant 0}\frac{\mathcal{G}_k(p)}{p}\int_\Omega\big|\tilde{T}^D\big|^2\,dx. \qquad (1)
\end{aligned}$$

Furthermore, we can easily see that $(\hat{\theta}_k, \partial_t\varepsilon_k^p T_k^D) \leqslant g\|\hat{\theta}_k\|^2 + \frac{g}{2}\|\hat{T}_k\|^2 + \frac{g}{2}\|\tilde{T}\|^2$ and so, finally

$$\begin{aligned}
&\partial_t\mathcal{E}\big(\hat{u}_k, \varepsilon_k^p, y_k\big)(t) + \frac{c\kappa}{\alpha}\|\nabla_x\hat{\theta}_k\|^2 + \frac{c}{2\alpha}\partial_t\|\hat{\theta}_k\|^2 + \frac{1}{2}\int_\Omega \frac{\mathcal{G}_k(\frac{\{|T_k^D|+\beta(\theta_k)\}_+}{y_k})}{|T_k^D|}\big|\hat{T}_k^D\big|^2\,dx \\
&\quad\leqslant \frac{1}{2}\bigg(\frac{1}{y_2}\sup_{p\geqslant 0}\frac{\mathcal{G}_k(p)}{p} + g\bigg)\big\|\tilde{T}^D\big\|^2 + \bigg(g + \frac{1}{2}\bigg)\|\hat{\theta}_k\|^2 + \frac{g}{2}\|\hat{T}_k\|^2 + \frac{c^2}{2}\|\operatorname{div}_x\partial_t\tilde{u}\|^2 \\
&\quad\leqslant \frac{1}{2}\bigg(\frac{1}{y_2}\sup_{p\geqslant 0}\frac{\mathcal{G}_k(p)}{p} + g\bigg)\big\|\tilde{T}^D\big\|^2 + \frac{c^2}{2}\|\operatorname{div}_x\partial_t\tilde{u}\|^2 + C\bigg(\frac{c}{2\alpha}\|\hat{\theta}_k\|^2 + \mathcal{E}\big(\hat{u}_k, \varepsilon_k^p, y_k\big)\bigg).
\end{aligned}$$

Applying Gronwall's lemma completes the proof of the desired inequality. The terms $\{\sup_{p\geqslant 0}\frac{\mathcal{G}_k(p)}{p}\}_k$ are uniformly bounded by the assumptions **A1**–**A2**. Moreover from Lemma 1 it follows that $\tilde{T} \in L^\infty(0, T^*; L^2(\Omega))$ and $\operatorname{div}_x\partial_t\tilde{u} \in L^\infty(0, T^*; L^2(\Omega))$ for all $T^* > 0$. Therefore the proof is completed. □

2.4. Estimates for time derivatives

Lemma 5. *For all $T^* \geqslant 0$ there exists a constant $C(T^*) \geqslant 0$ independent of k such that for all k the following inequality holds*

$$\begin{aligned}
&\int_0^t \mathcal{E}\big(\partial_t\hat{u}_k, \partial_t\varepsilon_k^p, \partial_t y_k\big)(\tau)\,d\tau + \int_0^t \big\|\partial_t\hat{\theta}_k(\tau)\big\|^2\,d\tau + \big\|\nabla_x\hat{\theta}_k(t)\big\|^2 \\
&\quad\leqslant C\big(T^*\big)\bigg(\int_0^t \big\|\partial_t\operatorname{div}_x\tilde{u}(\tau)\big\|^2\,d\tau + \int_0^t \big\|\hat{T}_k(\tau)\big\|^2\,d\tau + 1\bigg) + \big\|\nabla_x\hat{\theta}_k(0)\big\|^2.
\end{aligned}$$

Proof. After differentiation of Eqs. (GA1) and (GA2) with respect to t in the k-th step of the Galerkin approximation we proceed in the same way as in the previous lemma to obtain

$$\left(\partial_t \hat{T}_k, \partial_t \varepsilon(\nabla_x \hat{u}_k)\right) + \frac{c}{\alpha}\|\partial_t \hat{\theta}_k\|^2 + \frac{c\kappa}{2\alpha}\partial_t\|\nabla_x \hat{\theta}_k\|^2 = c(\partial_t \tilde{u}, \partial_t \nabla_x \hat{\theta}_k) + \frac{c}{\alpha}\left(\partial_t \varepsilon_k^p T_k^D, \partial_t \hat{\theta}_k\right).$$

Then we use the definition of the free energy to get

$$2\mathcal{E}\left(\partial_t \hat{u}_k, \partial_t \varepsilon_k^p, \partial_t y_k\right) + \frac{c}{\alpha}\|\partial_t \hat{\theta}_k\|^2 + \frac{c\kappa}{2\alpha}\partial_t\|\nabla_x \hat{\theta}_k\|^2 = c(\partial_t \tilde{u}, \partial_t \nabla_x \hat{\theta}_k) + \frac{c}{\alpha}\left(\partial_t \varepsilon_k^p \hat{T}_k, \partial_t \hat{\theta}_k\right) - \left(\partial_t \hat{T}_k, \partial_t \varepsilon_k^p\right).$$

We integrate by parts in the first term on the right-hand side, use the boundedness of $\partial_t \varepsilon_k^p$

$$\begin{aligned}
&2\mathcal{E}\left(\partial_t \hat{u}_k, \partial_t \varepsilon_k^p, \partial_t y_k\right) + \frac{c}{\alpha}\|\partial_t \hat{\theta}_k\|^2 + \frac{c\kappa}{2\alpha}\partial_t\|\nabla_x \hat{\theta}_k\|^2 \\
&\quad \leqslant -c(\partial_t \operatorname{div}_x \tilde{u}, \partial_t \hat{\theta}_k) + \left\|\partial_t \varepsilon_k^p\right\|_\infty \frac{c}{\alpha}\int_\Omega |\hat{T}_k||\partial_t \hat{\theta}_k|\,dx - \left(\partial_t \hat{T}_k, \partial_t \varepsilon_k^p\right),
\end{aligned}$$

and apply Young's inequality to every term on the right-hand side

$$\begin{aligned}
&2\mathcal{E}\left(\partial_t \hat{u}_k, \partial_t \varepsilon_k^p, \partial_t y_k\right) + \frac{c}{\alpha}\|\partial_t \hat{\theta}_k\|^2 + \frac{c\kappa}{2\alpha}\partial_t\|\nabla_x \hat{\theta}_k\|^2 \\
&\quad \leqslant \alpha c\|\partial_t \operatorname{div}_x \tilde{u}\|^2 + \frac{c}{4\alpha}\|\partial_t \hat{\theta}_k\|^2 + \frac{c}{\alpha}\left\|\partial_t \varepsilon_k^p\right\|_\infty^2 \|\hat{T}_k\|^2 + \frac{c}{4\alpha}\|\partial_t \hat{\theta}_k\|^2 + \epsilon\|\partial_t \hat{T}_k\|^2 + C(\epsilon)\left\|\partial_t \varepsilon_k^p\right\|^2,
\end{aligned}$$

where ϵ is sufficiently small to ensure the inequality $\epsilon\|\partial_t \hat{T}_k\|^2 \leqslant \mathcal{E}(\partial_t \hat{u}_k, \partial_t \varepsilon_k^p, \partial_t y_k)$. Thus we easily obtain

$$\mathcal{E}\left(\partial_t \hat{u}_k, \partial_t \varepsilon_k^p, \partial_t y_k\right) + \|\partial_t \hat{\theta}_k\|^2 + \partial_t\|\nabla_x \hat{\theta}_k\|^2 \leqslant C\left(\|\partial_t \operatorname{div}_x \tilde{u}\|^2 + g^2\|\hat{T}_k\|^2 + g^2\right).$$

Integration over time finishes the proof. □

2.5. Estimates for space derivatives

To show the estimates for space derivatives we need some propositions. In the first one we estimate the x-derivatives of functions y_k and ε_k^p. Then we estimate the second order space derivative of the function $\hat{u}_k$. Using the obtained inequality we get the estimate of $\nabla_x \hat{T}_k$ uniformly with respect to k.

Proposition 2. *For the approximate sequence $(\hat{u}_k, \varepsilon_k^p, y_k, \hat{\theta}_k)$ the following inequalities hold*

(i)

$$\left\|\partial_{x_i} y_k(t)\right\|^2 \leqslant \|\partial_{x_i} y_0\|^2 + C_1 \int_0^t \left(\left\|\partial_{x_i} T_k^D\right\|^2 + \|\partial_{x_i} y_k\|^2\right) d\tau,$$

(ii)

$$\left\|\partial_{x_i} \varepsilon_k^p(t)\right\|^2 \leqslant 2\left\|\partial_{x_i} \varepsilon_k^p(0)\right\| + tC_2 \int_0^t \left(\left\|\partial_{x_i} T_k^D\right\|^2 + \|\partial_{x_i} \theta_k\|^2 + \|\partial_{x_i} y_k\|^2\right) d\tau,$$

where $\theta_k = \hat{\theta}_k + \tilde{\theta}$,

(iii)

$$\left\|\hat{u}_k(t)\right\|_2^2 \leqslant C_3\left(\left\|\nabla_x \varepsilon_k^p\right\|^2 + \|\nabla_x \hat{\theta}_k\|^2\right),$$

and the positive constants C_1, C_2 and C_3 are independent of k.

Proof. The inequality (i) is proved in [5].

Using the approximated equation (GA3) and Jensen's inequality we easily obtain

$$\left\|\partial_{x_i} \varepsilon_k^p(t)\right\|^2 \leqslant 2t \int_0^t \left\|\partial_{x_i}\left(\mathcal{G}_k\left(\frac{\{|T_k^D| + \beta(\theta_k)\}_+}{y_k}\right) \cdot \frac{T_k^D}{T_k^D}\right)\right\|^2 d\tau + 2\left\|\partial_{x_i} \varepsilon_k^p(0)\right\|.$$

Then we calculate derivatives to get

$$\begin{aligned}\left\|\partial_{x_i}\varepsilon_k^p(t)\right\|^2 \leqslant 2t\int_0^t \Bigg\| &\mathcal{G}_k'\left(\frac{\{|T_k^D|+\beta(\theta_k)\}_+}{y_k}\right)\cdot\left(\frac{\partial_{x_i}|T_k^D|+\beta'(\theta_k)\partial_{x_i}\theta_k}{y_k}\chi_{\mathbf{Z}}-\frac{\{|T_k^D|+\beta(\theta_k)\}_+\partial_{x_i}y_k}{y_k^2}\right)\frac{T_k^D}{|T_k^D|}\\ &+\mathcal{G}_k\left(\frac{\{|T_k^D|+\beta(\theta_k)\}_+}{y_k}\right)\cdot\left(\frac{\partial_{x_i}T_k^D}{|T_k^D|}-\frac{T_k^D\partial_{x_i}|T_k^D|}{|T_k^D|^2}\right)\Bigg\|^2 d\tau\\ &+2\left\|\partial_{x_i}\varepsilon_k^p(0)\right\|\end{aligned}$$

where $\chi_{\mathbf{Z}}$ is the characteristic function of the set $\mathbf{Z}=\{|T_k^D|+\beta(\theta_k)>0\}$ (on $\Omega\setminus\mathbf{Z}$ we have $\partial_{x_i}\{|T_k^D|+\beta(\theta_k)\}_+=0$). Now it's easy to see that

$$\begin{aligned}\left\|\partial_{x_i}\varepsilon_k^p(t)\right\|^2 \leqslant{}& \left(4t\left(\sup_{p\geqslant 0}\mathcal{G}_k'(p)\frac{1}{y_2}\right)^2+8t\left(\sup_{p\geqslant 0}\frac{\mathcal{G}_k(p)}{p}\right)^2\right)\int_0^t\left\|\partial_{x_i}T_k^D\right\|^2 d\tau\\ &+4tB^2\left(\sup_{p\geqslant 0}\mathcal{G}_k'(p)\frac{1}{y_2}\right)^2\int_0^t\|\partial_{x_i}\theta_k\|^2 d\tau\\ &+4t\left(\sup_{p\geqslant 0}\mathcal{G}_k'(p)p\frac{1}{y_2}\right)^2\int_0^t\|\partial_{x_i}y_k\|^2 d\tau\\ &+2\left\|\partial_{x_i}\varepsilon_k^p(0)\right\|.\end{aligned}$$

The inequality above obviously completes the proof of (ii).

The inequality (iii) follows immediately from the ellipticity of the operator $u\mapsto \operatorname{div}_x\mathcal{D}(\varepsilon(\nabla_x u))$ and Eq. (GA1). □

Theorem 1. *There exists a constant $D(T^*)\geqslant 0$ independent of k such that for all $0\leqslant t\leqslant T^*$ and for all k the following inequality holds*

$$\begin{aligned}\left\|\nabla_x\hat{T}_k(t)\right\|^2 \leqslant D(T^*)\Bigg(&\|\nabla_x y_0\|^2+\left\|\nabla_x\varepsilon_k^p(0)\right\|^2+\mathcal{E}(\hat{u}_k,\varepsilon_k^p,y_k)(0)+\left\|\hat{\theta}_k^0\right\|^2\\ &+\int_0^t\|\partial_t\operatorname{div}_x\tilde{u}\|^2 d\tau+\int_0^t\|\nabla_x\tilde{\theta}\|^2 d\tau+\int_0^t\left\|\nabla_x\tilde{T}^D\right\|^2 d\tau\Bigg).\end{aligned}$$

Proof. Let us first consider the inequality (i) from Proposition 2. Using the Gronwall inequality one can show that

$$\left\|\partial_{x_i}y_k(t)\right\|^2 \leqslant \left(C_1te^{C_1t}+1\right)\left(\|\partial_{x_i}y_0\|^2+C_1\int_0^t\left\|\partial_{x_i}T_k^D\right\|^2 d\tau\right).$$

If we now add inequalities (i) and (ii) and use the inequality above we obtain for $\|\partial_{x_i}\varepsilon_k^p(t)\|$ the following estimate:

$$\begin{aligned}\left\|\partial_{x_i}\varepsilon_k^p(t)\right\|^2 &\leqslant \left\|\partial_{x_i}\varepsilon_k^p(t)\right\|^2+\left\|\partial_{x_i}y_k(\tau)\right\|^2\\ &\leqslant \left(te^{C_1t}(C_1+tC_2)+1\right)\|\partial_{x_i}y_0\|^2+2\left\|\partial_{x_i}\varepsilon_k^p(0)\right\|^2\\ &\quad+\left(C_1te^{C_1t}+1\right)(C_1+tC_2)\int_0^t\left\|\partial_{x_i}T_k^D\right\|^2 d\tau+tC_2\int_0^t\|\partial_{x_i}\theta_k\|^2 d\tau.\end{aligned}$$

Using Eq. (GA2) differentiated with respect to x we get

$$\left\|\nabla_x\hat{T}_k(t)\right\|^2 \leqslant D_1\left(\left\|\hat{u}_k(t)\right\|_2^2+\left\|\nabla_x\varepsilon_k^p(t)\right\|^2\right).$$

The inequality (iii) from Proposition 2 and the estimate for $\|\partial_{x_i}\varepsilon_k^p(t)\|$, shown above, give

$$\left\|\nabla_x \hat{T}_k(t)\right\|^2 \leqslant D_1(t)\Bigg(\|\nabla_x y_0\|^2 + \left\|\nabla_x \varepsilon_k^p(0)\right\|^2 + \int_0^t \|\nabla_x \hat{T}_k\|^2\,d\tau + \int_0^t \left\|\nabla_x \tilde{T}^D\right\|^2 d\tau$$
$$+ \left\|\nabla_x \hat{\theta}_k(t)\right\|^2 + \int_0^t \|\nabla_x \hat{\theta}_k\|^2\,d\tau + \int_0^t \|\nabla_x \tilde{\theta}\|^2\,d\tau\Bigg).$$

Lemmas 4 and 5 yield estimates for $\|\nabla_x \hat{\theta}_k(t)\|^2$ and $\int_0^t \|\nabla_x \hat{\theta}_k\|^2\,d\tau$, which imply

$$\left\|\nabla_x \hat{T}_k(t)\right\|^2 \leqslant D_2(t)\Bigg(\|\nabla_x y_0\|^2 + \left\|\nabla_x \varepsilon_k^p(0)\right\|^2 + \mathcal{E}\big(\hat{u}_k, \varepsilon_k^p, y_k\big)(0) + \left\|\hat{\theta}_k^0\right\|^2$$
$$+ \int_0^t \|\partial_t \operatorname{div}_x \tilde{u}\|^2\,d\tau + \int_0^t \|\nabla_x \tilde{\theta}\|^2\,d\tau + \int_0^t \left\|\nabla_x \tilde{T}^D\right\|^2 d\tau + \int_0^t \|\nabla_x \hat{T}_k\|^2\,d\tau\Bigg),$$

where the constant D_2 does not depend on k. A further use of Gronwall's inequality completes the proof. $\square$

We have shown all the energy estimates that are needed to prove the existence of a solution to (HP).

2.6. The existence of the solution to (HP) and (TBP)

Theorem 2. *Assume that the initial data satisfy*:

$$\varepsilon_0^p \in H^1(\Omega), \qquad y_0 \in H^1(\Omega), \qquad \hat{\theta}_0 \in H^1(\Omega),$$

with ε_0^p symmetric and with $\operatorname{tr} \varepsilon_0^p(x) = 0$ for almost all $x \in \Omega$, the initial data y_0 satisfy $y_2 \leqslant y_0(x) \leqslant y_1$ for almost all $x \in \Omega$, and the boundary data $g_{\mathcal{D}}$, g_θ and external force F have regularity as in Lemma 1. *Moreover suppose that assumptions* **A1**–**A11** *hold. Then there exists a solution $(\hat{u}, \varepsilon^p, y, \hat{\theta})$ of the problem* (HP) *global in time such that*

$$\hat{u} \in L^\infty\big(0, T^*; H^2(\Omega)\big), \qquad \partial_t \hat{u} \in L^2\big(0, T^*; H^1(\Omega)\big),$$
$$\varepsilon^p \in L^\infty\big(0, T^*; H^1(\Omega)\big), \qquad \partial_t \varepsilon^p \in L^\infty\big(0, T^*; L^\infty(\Omega)\big),$$
$$y \in L^\infty\big(0, T^*; H^1(\Omega)\big), \qquad \partial_t y \in L^\infty\big(0, T^*; L^2(\Omega)\big),$$
$$\hat{\theta} \in L^2\big(0, T^*; H^2(\Omega)\big) \cap L^\infty\big(0, T^*; H_0^1(\Omega)\big), \qquad \partial_t \hat{\theta} \in L^2\big(0, T^*; L^2(\Omega)\big),$$

for all $T^* > 0$. Moreover the function y is bounded on $(0, T^*) \times \Omega$ and satisfies $y_2 \leqslant y(x) \leqslant y_1$ for almost all $x \in \Omega$.

Proof. We will proceed quite similarly as in the proof of Theorem 2.9 in [5]. Thus first we approximate the given data $(\varepsilon_0^p, y^0, \hat{\theta}^0, F, g_{\mathcal{D}}, g_\theta)$ by a smooth sequence $(\varepsilon_{0,i}^p, y_i^0, \hat{\theta}_i^0, F_i, g_{\mathcal{D},i}, g_{\theta,i})$, while convergence is meant in the topology above. For data $(\varepsilon_{0,i}^p, y_i^0, \hat{\theta}_i^0, F_i, g_{\mathcal{D},i}, g_{\theta,i})$ we construct the Galerkin sequence $(\hat{u}_{ik}, \varepsilon_{ik}^p, y_{ik}, \hat{\theta}_{ik})$ to approximate the solution of (HP) by smooth functions. Using the estimates proved in Lemma 4, Lemma 5 and Theorem 1 we obtain that for all $T^* > 0$ the approximate sequence satisfies:

$$\|\hat{u}_{ik}\|_{L^\infty(0,T^*;H^2(\Omega))}, \qquad \|\partial_t \hat{u}_{ik}\|_{L^2(0,T^*;H^1(\Omega))},$$
$$\left\|\varepsilon_{ik}^p\right\|_{L^\infty(0,T^*;H^1(\Omega))}, \qquad \left\|\partial_t \varepsilon_{ik}^p\right\|_{L^\infty(0,T^*;L^\infty(\Omega))},$$
$$\|y_{ik}\|_{L^\infty(0,T^*;H^1(\Omega))}, \qquad \|\partial_t y_{ik}\|_{L^\infty(0,T^*;L^2(\Omega))},$$
$$\|\hat{\theta}_{ik}\|_{L^\infty(0,T^*;H_0^1(\Omega))}, \qquad \|\partial_t \hat{\theta}_{ik}\|_{L^2(0,T^*;L^2(\Omega))} \leqslant M\big(T^*\big)$$

for some constant $M(T^*) \geqslant 0$ independent of the approximation step. Thus we can select a subsequence, which is weakly convergent (weakly-$*$ convergent in the case of L^∞ space) to $(\hat{u}, \varepsilon^p, y, \hat{\theta})$. By Rellich–Kondrachov theorem the imbending $H^1((0, T^*) \times \Omega) \subset L^2((0, T^*) \times \Omega)$ is compact, therefore the sequence $(\hat{u}_{ik}, \varepsilon_{ik}^p, y_{ik}, \hat{\theta}_{ik})$ is precompact in $L^2((0, T^*) \times \Omega)$ and we can choose a subsequence which converges strongly in $L^2((0, T^*) \times \Omega)$ to the same limit $(\hat{u}, \varepsilon^p, y, \hat{\theta})$.

We will prove now that limit functions $(\hat{u}, \varepsilon^p, y, \hat{\theta})$ solve (HP) and for simplicity we will denote the subsequence chosen before by $(\hat{u}_k, \varepsilon_k^p, y_k, \hat{\theta}_k)$. It is easy to see that $(\hat{u}, \varepsilon^p, y, \hat{\theta})$ satisfies Eqs. (HP1), (HP2). To see that the nonlinear equation (HP3) is satisfied we first check that the sequence of arguments of the functions $\mathcal{G}_k$ converges strongly in $L^2((0, T^*) \times \Omega)$. Indeed, we have $T_k^D \to T^D$ in $L^2((0, T^*) \times \Omega)$. Moreover,

$$\int\limits_{(0,T^*)\times\Omega} \big|\beta(\tilde\theta+\hat\theta_k)-\beta(\tilde\theta+\hat\theta)\big|^2\,dt\,dx \leqslant B \int\limits_{(0,T^*)\times\Omega} |\hat\theta_k-\hat\theta|^2\,dt\,dx \to 0, \quad \text{when } k\to 0,$$

implies $\{|T_k^D|+\beta(\theta_k)\}_+ \to \{|T^D|+\beta(\theta)\}_+$ in $L^2((0,T^*)\times\Omega)$. Furthermore, using Lemma 3 we conclude that

$$\begin{aligned}
&\int\limits_{(0,T^*)\times\Omega} \left|\frac{\{|T_k^D|+\beta(\theta_k)\}_+}{y_k} - \frac{\{|T^D|+\beta(\theta)\}_+}{y}\right|^2 dt\,dx \\
&\quad\leqslant \frac{1}{y_2^2}\int\limits_{(0,T^*)\times\Omega} \big|\big\{|T_k^D|+\beta(\theta_k)\big\}_+ - \big\{|T^D|+\beta(\theta)\big\}_+\big|^2\,dt\,dx \\
&\qquad + \frac{1}{y_2^4}\int\limits_{(0,T^*)\times\Omega} \big|\big\{|T^D|+\beta(\theta)\big\}_+\big|^2 |y_k-y|^2\,dt\,dx \to 0, \quad \text{when } k\to 0.
\end{aligned}$$

The convergence of the second term on the right-hand side of the inequality above follows from Lebesgue's dominated convergence theorem. Now let's estimate

$$\begin{aligned}
&\int\limits_{(0,T^*)\times\Omega} \left|\mathcal{G}_k\left(\frac{\{|T_k^D|+\beta(\theta_k)\}_+}{y_k}\right) - \mathcal{G}\left(\frac{\{|T^D|+\beta(\theta)\}_+}{y}\right)\right|^2 dt\,dx \\
&\quad\leqslant \int\limits_{(0,T^*)\times\Omega} \left|\mathcal{G}_k\left(\frac{\{|T_k^D|+\beta(\theta_k)\}_+}{y_k}\right) - \mathcal{G}\left(\frac{\{|T_k^D|+\beta(\theta_k)\}_+}{y_k}\right)\right|^2 dt\,dx \\
&\qquad + \int\limits_{(0,T^*)\times\Omega} \left|\mathcal{G}\left(\frac{\{|T_k^D|+\beta(\theta_k)\}_+}{y_k}\right) - \mathcal{G}\left(\frac{\{|T^D|+\beta(\theta)\}_+}{y}\right)\right|^2 dt\,dx \to 0, \quad \text{when } k\to 0.
\end{aligned}$$

The first term on the right-hand side of the inequality above converges because of the uniform convergence $\mathcal{G}_k \rightrightarrows \mathcal{G}$, convergence of the second term is a consequence of Lebesgue's dominated convergence theorem. Further denoting by $G_k := \mathcal{G}_k(\frac{\{|T_k^D|+\beta(\theta_k)\}_+}{y_k})$ and $G := \mathcal{G}(\frac{\{|T^D|+\beta(\theta)\}_+}{y})$ we have the estimate

$$\begin{aligned}
&\int\limits_{(0,T^*)\times\Omega} \left|G_k\frac{T_k^D}{|T_k^D|} - G\frac{T^D}{|T^D|}\right|^2 dt\,dx \\
&\quad\leqslant \int\limits_{(0,T^*)\times\Omega} |G_k-G|^2\,dt\,dx + \int\limits_{(0,T^*)\times\Omega} \left|G\frac{T_k^D}{|T_k^D|} - G\frac{T^D}{|T^D|}\right|^2 dt\,dx \to 0, \quad \text{when } k\to 0
\end{aligned}$$

(we have once more used Lebesgue's dominated convergence theorem). The inequality above gives us that $(\hat u, \varepsilon^p, y, \hat\theta)$ satisfies (HP3). Similarly we can show that $(\hat u, \varepsilon^p, y, \hat\theta)$ satisfies (HP4). To prove that Eq. (HP5) is satisfied it is enough to observe that $\partial_t\varepsilon_{ik}^p \cdot \hat T_k \to \partial_t\varepsilon^p \cdot \hat T$ strongly in $L^2(0, T^*\times\Omega)$ when $k\to\infty$. This is an easy consequence of the convergences proven above.

Moreover from Lemma 3 we get the required estimate for y. Additionally $\hat\theta \in L^2(0,T^*;H^2(\Omega))$ because this function satisfies the parabolic equation (HP5) with the right-hand side of regularity $L^2((0,T^*)\times\Omega)$. □

Now the existence of the solution to (TBP) is an easy corollary from Lemma 1 and the theorem above.

Corollary 1. *Assume that the initial data satisfy*:

$$\varepsilon_0^p \in H^1(\Omega), \qquad y_0 \in H^1(\Omega), \qquad \theta_0 \in H^1(\Omega),$$

with ε_0^p symmetric and with $\operatorname{tr}\varepsilon_0^p(x)=0$ for almost all $x\in\Omega$, the initial data y_0 satisfy $y_2\leqslant y_0(x)\leqslant y_1$ for almost all $x\in\Omega$, and the boundary data $g_{\mathcal{D}}$, g_θ and external force f have regularity as in Lemma 1. *Let the boundary and the initial data g_θ and θ_0 satisfy the compatibility condition*

$$\theta_0(x) = g_\theta(x) \quad \text{for } x\in\partial\Omega.$$

Moreover suppose that the assumptions **A1–A11** *hold. Then there exists a solution global in time $(u,\varepsilon^p,y,\theta)$ of the problem* (TBP) *such that following estimates hold*

$$
\begin{aligned}
&u \in L^\infty\big(0,T^*;H^2(\Omega)\big), \qquad \partial_t u \in L^2\big(0,T^*;H^1(\Omega)\big),\\
&\varepsilon^p \in L^\infty\big(0,T^*;H^1(\Omega)\big), \qquad \partial_t\varepsilon^p \in L^\infty\big(0,T^*;L^\infty(\Omega)\big),\\
&y \in L^\infty\big(0,T^*;H^1(\Omega)\big), \qquad \partial_t y \in L^\infty\big(0,T^*;L^2(\Omega)\big),\\
&\theta \in L^2\big(0,T^*;H^2(\Omega)\big)\cap L^\infty\big(0,T^*;H^1(\Omega)\big), \qquad \partial_t\theta \in L^2\big(0,T^*;L^2(\Omega)\big),
\end{aligned}
$$

for all $T^*>0$. Moreover the function y is bounded on $(0,T^*)\times\Omega$ and satisfies $y_2 \leqslant y(x) \leqslant y_1$ for almost all $x\in\Omega$.

3. Uniqueness of the solution to (TBP)

We are going to prove the uniqueness result in the manner of [5]. Let us define $(u^\Delta,\varepsilon^{p,\Delta},y^\Delta,\theta^\Delta) := (u^1-u^2,\varepsilon^{p,1}-\varepsilon^{p,2},y^1-y^2,\theta^1-\theta^2)$, where $(u^i,\varepsilon^{p,i},y^i,\theta^i)$ are the solutions of (TBP) corresponding to $A:=A^i$ in (TBP4). Thus they satisfy the following system of equations

$$
\begin{aligned}
&\operatorname{div}_x T^\Delta(x,t) = c\nabla_x\theta^\Delta(x,t),\\
&T^\Delta(x,t) = \mathcal{D}\big(\varepsilon\big(\nabla_x u(x,t)\big)-\varepsilon^p(x,t)\big),\\
&\partial_t\varepsilon^{p,\Delta}(x,t) = \mathcal{G}\bigg(\frac{\{|T^{D,1}(x,t)|+\beta(\theta^1(x,t))\}_+}{y^1(x,t)}\bigg)\frac{T^{D,1}(x,t)}{|T^{D,1}(x,t)|}\\
&\qquad\qquad - \mathcal{G}\bigg(\frac{\{|T^{D,2}(x,t)|+\beta(\theta^2(x,t))\}_+}{y^2(x,t)}\bigg)\frac{T^{D,2}(x,t)}{|T^{D,2}(x,t)|},\\
&\partial_t y^\Delta(x,t) = \gamma\big(y^1(x,t)\big)\mathcal{G}\bigg(\frac{|T^{D,1}(x,t)|}{y^1(x,t)}\bigg)\big|T^{D,1}(x,t)\big| - A^1\delta\big(y^1(x,t)\big)\\
&\qquad\qquad - \gamma\big(y^2(x,t)\big)\mathcal{G}\bigg(\frac{|T^{D,2}(x,t)|}{y^2(x,t)}\bigg)\big|T^{D,2}(x,t)\big| + A^2\delta\big(y^2(x,t)\big),\\
&\partial_t\theta^\Delta(x,t) = \kappa\Delta\theta^\Delta(x,t) - \alpha\operatorname{div}_x\partial_t u^\Delta(x,t) + \partial_t\varepsilon^{p,1}(x,t)\cdot T^{D,1}(x,t) - \partial_t\varepsilon^{p,2}(x,t)\cdot T^{D,2}(x,t),
\end{aligned}
$$

with homogeneous initial data and boundary conditions. Using energy estimates analogous to Lemma 4 we can observe that the norm of the differences $(u^\Delta,\varepsilon^{p,\Delta},y^\Delta,\theta^\Delta)$ is estimated by $|A^1-A^2|$ and we can formulate a result similar to Theorem 3.1 from [5]. Unfortunately to prove the following theorem we need a more restrictive version of the assumption **A4**:

A4′ There exists g' such, that for all $p\in\mathbb{R}_+\mathcal{G}'(p)p^2<g'$.

Remark 8. It is easy so see that the function $\mathcal{G}$ proposed by S.R. Bodner and Y. Partom in [1] satisfies the assumption **A4′**.

Theorem 3. *Let $(u^1,\varepsilon^{p,1},y^1,\theta^1)$ and $(u^2,\varepsilon^{p,2},y^2,\theta^2)$ be the solutions of* (TBP) *with the same initial data, boundary condition, and external force, related to $A^1, A^2 < A^*$ respectively. Moreover let for almost all $(x,t)\in\Omega\times(0,\infty)$ hold $y_2\leqslant y^i\leqslant y_1$ $(i=1,2)$. Then the difference $(u^\Delta,\varepsilon^{p,\Delta},y^\Delta,\theta^\Delta)$ satisfies*

$$
\mathcal{E}\big(u^\Delta,\varepsilon^{p,\Delta},y^\Delta,\theta^\Delta\big)(t) + \big\|\theta^\Delta(t)\big\|^2 + \big\|y^\Delta(t)\big\|^2 \leqslant M_1\exp(M_2 t)\big|A^1-A^2\big|^2
$$

where M_1 does not depend on A^* and M_2 depends affinely on A^*.

Proof. Using the same methods as in Lemma 4 we obtain

$$
\begin{aligned}
&\partial_t\mathcal{E}\big(u^\Delta,\varepsilon^{p,\Delta},y^\Delta,\theta^\Delta\big)(t) + \frac{c}{2\alpha}\partial_t\big\|\theta^\Delta(t)\big\|^2 + \frac{c\kappa}{\alpha}\big\|\nabla_x\theta^\Delta(t)\big\|^2\\
&\quad = \frac{c}{\alpha}\big(\theta^\Delta(t),\partial_t\varepsilon^{p,1}(t)\cdot T^{D,1}(t) - \partial_t\varepsilon^{p,2}(t)\cdot T^{D,2}(t)\big) - \big(T^\Delta(t),\partial_t\varepsilon^{p,\Delta}(t)\big).
\end{aligned}
$$

In the same way as in [5] we get that

$$
-\big(T^\Delta,\partial_t\varepsilon^{p,\Delta}\big) \leqslant -\int_\Omega\bigg(\mathcal{G}\bigg(\frac{\{|T^{D,1}|+\beta(\theta^1)\}_+}{y^1}\bigg) - \mathcal{G}\bigg(\frac{\{|T^{D,2}|+\beta(\theta^2)\}_+}{y^2}\bigg)\bigg)\big(\big|T^{D,1}\big|-\big|T^{D,2}\big|\big)\,dx,
$$

then also in the manner of [5] using assumptions **A1** and **A2** we obtain that there exists a constant C such that $\mathcal{G}'(p)p+\mathcal{G}'(p)\leqslant C$ for all $p\geqslant 0$ and therefore

$$\begin{aligned}
&\left|\mathcal{G}\left(\frac{\{|T^{D,1}|+\beta(\theta^1)\}_+}{y^1}\right)-\mathcal{G}\left(\frac{\{|T^{D,2}|+\beta(\theta^2)\}_+}{y^2}\right)\right| \\
&\quad\leqslant \frac{C}{1+\min\{\frac{\{|T^{D,1}|+\beta(\theta^1)\}_+}{y^1},\frac{\{|T^{D,2}|+\beta(\theta^2)\}_+}{y^2}\}}\left[\max\left\{\frac{1}{y^1},\frac{1}{y^2}\right\}\left|\left\{\left|T^{D,1}\right|+\beta\left(\theta^1\right)\right\}_+-\left\{\left|T^{D,2}\right|+\beta\left(\theta^2\right)\right\}_+\right|\right. \\
&\qquad\left.+\min\left\{\left\{\left|T^{D,1}\right|+\beta\left(\theta^1\right)\right\}_+,\left\{\left|T^{D,2}\right|+\beta\left(\theta^2\right)\right\}_+\right\}\left|\frac{1}{y^1}-\frac{1}{y^2}\right|\right] \\
&\quad\leqslant \frac{C}{y_2}\left(\left|T^{D,1}-T^{D,2}\right|+B\left|\theta^1-\theta^2\right|+\frac{y_1}{y_2}\left|y^1-y^2\right|\right).
\end{aligned}$$

Above, we have also used the inequality $|\beta(\theta^1)-\beta(\theta^2)|\leqslant B|\theta^1-\theta^2|$ which follows from assumptions **A8** and **A11**. Now we estimate the term with the difference of temperature functions. If we use the equation for $\partial_t\varepsilon^{p,\Delta}$ we obtain

$$\begin{aligned}
&\left(\theta^\Delta(t),\partial_t\varepsilon^{p,1}(t)\cdot T^{D,1}(t)-\partial_t\varepsilon^{p,2}(t)\cdot T^{D,2}(t)\right) \\
&\quad\leqslant \int_\Omega\left|\theta^\Delta(t)\right|\left|\mathcal{G}\left(\frac{\{|T^{D,1}|+\beta(\theta^1)\}_+}{y^1}\right)\left|T^{D,1}\right|-\mathcal{G}\left(\frac{\{|T^{D,2}|+\beta(\theta^2)\}_+}{y^2}\right)\left|T^{D,2}\right|\right|dx.
\end{aligned}$$

Thus we easily obtain that

$$-\left(T^\Delta,\partial_t\varepsilon^{p,\Delta}\right)\leqslant C_1\left\|T^{D,\Delta}\right\|^2+C_2\left\|\theta^\Delta\right\|^2+C_3\left\|y^\Delta\right\|^2,$$

where C_1, C_2 and C_3 do not depend on A^*. Next, similarly as in [5] using the assumption **A4′**, **A8**, **A11** we can estimate

$$\begin{aligned}
&\left|\mathcal{G}\left(\frac{\{|T^{D,1}|+\beta(\theta^1)\}_+}{y^1}\right)\left|T^{D,1}\right|-\mathcal{G}\left(\frac{\{|T^{D,2}|+\beta(\theta^2)\}_+}{y^2}\right)\left|T^{D,2}\right|\right| \\
&\quad\leqslant\left(\sup_{p\geqslant 0}\mathcal{G}(p)+\frac{\bar{C}y_1}{(y_2)^2}\right)\left(\left|T^{D,1}-T^{D,2}\right|+B\left|\theta^1-\theta^2\right|\right)+\frac{\bar{C}(y_1)^2}{(y_2)^2}\left|y^1-y^2\right|,
\end{aligned}$$

where $\mathcal{G}'(p)p^2+\mathcal{G}'(p)\leqslant\bar{C}$ for all $p\geqslant 0$. In the last part of the proof we use the following inequality:

$$\partial_t\left\|y^\Delta\right\|^2\leqslant D_1\left(\left\|y^\Delta\right\|^2+\left\|T^{D,\Delta}\right\|^2\right)+D_2\left|A^1-A^2\right|^2$$

proved in [5] (using again the stronger assumption **A4′**), where D_1 depends affinely on A^* and D_2 does not depend on A^*. Finally we obtain

$$\begin{aligned}
&\partial_t\mathcal{E}\left(u^\Delta,\varepsilon^{p,\Delta},y^\Delta,\theta^\Delta\right)(t)+\partial_t\left\|\theta^\Delta(t)\right\|^2+\left\|\nabla_x\theta^\Delta(t)\right\|^2+\partial_t\left\|y^\Delta(t)\right\|^2 \\
&\quad\leqslant M\left(\mathcal{E}\left(u^\Delta,\varepsilon^{p,\Delta},y^\Delta,\theta^\Delta\right)(t)+\left\|\theta^\Delta(t)\right\|^2+\left\|y^\Delta(t)\right\|^2\right)+D_2\left|A^1-A^2\right|^2.
\end{aligned}$$

The proof is completed by applying Gronwall's inequality. □

Corollary 2 *(Uniqueness). Under assumptions of Theorem* 3 *if $A^1=A^2=A$ then for all $t>0$ and for almost all $x\in\Omega$ holds*

(i) $T^1(x,t)=T^2(x,t)$,
(ii) $\theta^1(x,t)=\theta^2(x,t)$,
(iii) $y^1(x,t)=y^2(x,t)$,
(iv) $\varepsilon^{p,1}(x,t)=\varepsilon^{p,2}(x,t)$.

Proof. One sees that (i)–(iii) immediately follow from Theorem 3. Using the equation on $\partial_t\varepsilon^{p,\Delta}$ multiplied by $\varepsilon^{p,\Delta}$ and integrated we get that

$$\frac{1}{2}\partial_t\left\|\varepsilon^{p,\Delta}\right\|^2\leqslant\int_\Omega\left|\mathcal{G}\left(\frac{\{|T^{D,1}|+\beta(\theta^1)\}_+}{y^1}\right)\frac{T^{D,1}}{|T^{D,1}|}-\mathcal{G}\left(\frac{\{|T^{D,2}|+\beta(\theta^2)\}_+}{y^2}\right)\frac{T^{D,2}}{|T^{D,2}|}\right|\left|\varepsilon^{p,\Delta}\right|dx.$$

Using identities (i)–(iii) and the continuity of the function $\mathcal{G}$ we obtain

$$\partial_t\left\|\varepsilon^{p,\Delta}\right\|^2\leqslant 0$$

and therefore (iv) holds. □

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